

Exam I, MTH 221, Fall 2010, 2pm section

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QUESTION 1. (30 points) Given $A^{-1} = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 3 & 0 \\ -2 & 2 & 3 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & -2 & -2 \\ -4 & 2 & 2 \\ 0 & 1 & -2 \end{bmatrix}$

(i) Find the 2nd row of $(AB)^{-1} = B^{-1}A^{-1}$

$$-4 \begin{bmatrix} 2 & -2 & -2 \end{bmatrix} + 2 \begin{bmatrix} -2 & 3 & 0 \end{bmatrix} + -2 \begin{bmatrix} -2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -8 & 10 & 2 \end{bmatrix}$$

(ii) Find the third column of $(BA)^{-1} = A^{-1}B^{-1}$

$$-2 \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} + -2 \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 10 \\ 2 \end{bmatrix}$$

(iii) Find the (2, 3)-entry of $B^{-1}A^{-1} \Rightarrow$ 2nd row of B^{-1} , 3rd col. of A^{-1}

$$\begin{bmatrix} -4 & 2 & 2 \end{bmatrix} \times \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} = (-4)(-2) + 2(0) + 2(3) = 8 + 6 = 14$$

(iv) Solve the system $AX = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$
given A^{-1}

$$A^{-1}AX = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 3 & 0 \\ -2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$$

$$I_3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 3 & 0 \\ -2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$$

3x3 3x1

\rightarrow we'll get 3x1

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} + -1 \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -1 \end{bmatrix}$$

$$x_1 = 4$$

$$x_2 = -11$$

$$x_3 = -1$$

(v) Find the matrix $A \Rightarrow (A^{-1})^{-1} = A$
3x3

$$A^{-1} = \begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & 0 \\ -2 & 2 & 3 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|ccc} 2 & -2 & -2 & 1 & 0 & 0 \\ -2 & 2 & 0 & 0 & 1 & 0 \\ -2 & 2 & 3 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & \frac{1}{2} & 0 & 0 \\ -2 & 2 & 0 & 0 & 1 & 0 \\ -2 & 2 & 3 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\begin{array}{l} 2R_1 + R_2 \rightarrow R_2 \\ 2R_1 + R_3 \rightarrow R_3 \end{array}}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array} \right] \xrightarrow{\begin{array}{l} 2R_2 + R_3 \rightarrow R_2 \\ 3R_3 + R_1 \rightarrow R_1 \end{array}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & \frac{3}{2} & 1 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array} \right] \xrightarrow{\begin{array}{l} 2R_2 + R_3 \rightarrow R_2 \\ 3R_3 + R_1 \rightarrow R_1 \end{array}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{9}{2} & 1 & 3 \\ 0 & 1 & 0 & 3 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{9}{2} & 1 & 3 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 3 & 1 & 2 \end{array} \right] \xrightarrow{\begin{array}{l} R_1 + R_2 \rightarrow R_1 \\ -\frac{1}{2}R_1 + R_2 \rightarrow R_2 \end{array}}$$

has $-\det(D)$

$$A = \begin{bmatrix} \frac{9}{2} & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

QUESTION 2. (12 points) Let $D = \begin{bmatrix} 4 & 2 & b & a \\ -4 & -2 & -2 & 2 \\ 2 & 2b & 0 & 1 \\ 0 & 0 & 0 & 10 \end{bmatrix}$
4x4

(i) For what values of a, b does the system $DX = \begin{bmatrix} 5 \\ -5 \\ 7 \\ 0 \end{bmatrix}$ have a unique solution?

$$\begin{bmatrix} 4 & 2 & b & a \\ 0 & 2b-1 & -2b & -a+1 \\ 0 & 0 & b-2 & a-1 \\ 0 & 0 & 0 & 10 \end{bmatrix} \Rightarrow \det = 4(2b-1)(b-2)(10)$$

Set it at 0 $\Rightarrow 4(2b-1)(b-2)(10) = 0$

$$2b-1 = 0 \quad b-2 = 0$$

$$b = \frac{1}{2} \quad b = 2$$

b is any real no except for $\frac{1}{2}$ or 2

a can be any real no. since it doesn't lie on the main diagonal so it doesn't affect the determinant

(ii) For what values of a, b will D be singular (non-invertible)? $\Rightarrow \det(D) = 0$

To have a determinant of 0, $b = \frac{1}{2}$, $b = 2$ and/or $a =$ any real number

QUESTION 3. (30 points) Let A be a 4×4 matrix. Given

$$A \xrightarrow{3R_1} A_1 \xrightarrow{R_3 \leftrightarrow R_2} A_2 \xrightarrow{-5R_2} A_3 \xrightarrow{-4R_1 + R_2 \rightarrow R_2} A_4 = \begin{bmatrix} -4 & 2 & 2 & -4 \\ 0 & 5 & 3 & -2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

has $3\det(A)$ has $15\det(A)$
 has $-3\det(A)$ has $15\det(A)$

$$\det(A_4) = (-4)(5)(3)(2) = -120$$

(i) Find $\det(A)$.

$$\det(A_4) = 15\det(A)$$

$$-120 = 15\det(A) \Rightarrow \det(A) = -8$$

(ii) Find $\det(-2A_2) = (-2)^4 \det(A_2) = 16(-3\det(A)) = 16(-3)(-8) = 384$

(iii) Find a matrix B such that $BA = A_2$

4x4 use I_4

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{3R_1} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_2} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = B$$

$$\therefore \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A = A_2$$

(iv) Find $\det(A^2 A_2) = \det(A A A_2) = \det(A)\det(A)\det(A_2) = (-8)(-8)(-3)(-8) = 1536$

(v) Find $\det((A^{-1} A_4)^T) = \det((A_4^T A^{-1})^T) = \det(A_4^T) \det(A^{-1}) = \det(A_4) \det(A^{-1})$
 $= (-120) \left(\frac{1}{\det(A)} \right) = (-120) \left(\frac{1}{-8} \right) = 15$

(vi) Find the (3,4)-entry of A_4^{-1}

let $C = A_4$

$$= \frac{C_{43}}{\det(A_4)} = \frac{(-1)^7 |C_{43}|}{\det(A_4)} = - \frac{\begin{vmatrix} -4 & 2 & -4 \\ 0 & 5 & -2 \\ 0 & 0 & 2 \end{vmatrix}}{-120}$$

sp. case

$$= \frac{\begin{vmatrix} -4 & 2 & -4 & -4 & 2 \\ 0 & 5 & -2 & 0 & 5 \\ 0 & 0 & 2 & 0 & 0 \end{vmatrix}}{-120} = \frac{-[(-40) + 0 + 0] - [0 + 0 + 0]}{-120} = \frac{40}{-120} = -\frac{1}{3}$$

QUESTION 4. (12 points) (b) Find a 2×2 matrix A such that $\begin{bmatrix} 7 & 7 \\ 1 & 5 \end{bmatrix} A + 3I_2 = 3A + \begin{bmatrix} 4 & 2 \\ 0 & 2 \end{bmatrix}$

$$\begin{aligned} \begin{bmatrix} 7 & 7 \\ 1 & 5 \end{bmatrix} A - 3A &= \begin{bmatrix} 4 & 2 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \\ \begin{bmatrix} 4 & 4 \\ 1 & 2 \end{bmatrix} A - 3A &= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \\ \begin{bmatrix} 7 & 7 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} &= \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix} \\ \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix} A &= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix} \Rightarrow \text{its inv} &= \frac{1}{-1} \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix} \\ \det \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix} &= 4(2) - 7(1) = -1 \\ \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix} A &= \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \\ \text{1st col. } 1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} -7 \\ 4 \end{bmatrix} &= \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ \text{2nd col. } 2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + -1 \begin{bmatrix} -7 \\ 4 \end{bmatrix} &= \begin{bmatrix} 4 \\ -2 \end{bmatrix} \end{aligned}$$

QUESTION 5. (16 points) Solve the following system:

$$\begin{aligned} x_3 + x_4 - x_5 &= 1 \\ x_1 + x_2 - x_3 + x_5 &= 2 \\ 2x_1 + 3x_2 + x_3 - 2x_5 &= 10 \end{aligned}$$

Then give me one numerical solution to the system.

$$\begin{aligned} \begin{bmatrix} 0 & 0 & 1 & 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 & 2 \\ 2 & 3 & 1 & 0 & -2 & 0 & 10 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2, 3R_1 - R_2} \begin{bmatrix} 1 & 1 & -1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 3 & 0 & -4 & 0 & 6 \end{bmatrix} \xrightarrow{R_2 + R_1 - 3R_3} \begin{bmatrix} 1 & 1 & -1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 & 3 \\ 0 & 1 & 3 & 0 & -4 & 0 & 6 \end{bmatrix} \xrightarrow{-2R_2 + R_3 \rightarrow R_3} \begin{bmatrix} 1 & 1 & -1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 3 & -1 & -4 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 0 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 & 1 & 0 & 3 \end{bmatrix} \xrightarrow{R_3 + R_2 - 3R_1} \begin{bmatrix} 0 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 & 1 & 0 & 3 \end{bmatrix}$$

x_1	x_2	x_3	x_4	x_5	C
0	0	1	1	-1	0
0	1	0	1	0	3
0	1	0	-3	1	3

stop

READ:

$$\begin{aligned} x_3 + x_4 - x_5 &= 1 \\ x_1 + 4x_4 + x_5 &= 0 \\ x_2 - 3x_4 - x_5 &= 3 \end{aligned}$$

$$\begin{aligned} x_4, x_5 &\in \mathbb{R} \\ x_4 = 0 \quad x_5 = 1, &\text{ then} \\ x_3 &= 1 + 1 - 0 = 2 \\ x_1 &= -4(0) - 1 = -1 \\ x_2 &= 3 + 3(0) + 1 = 4 \end{aligned}$$

x_1, x_2, x_3 are leading variables
 x_4, x_5 are free variables

$$\begin{aligned} x_3 &= 1 + x_4 - x_5 \\ x_1 &= -4x_4 - x_5 \\ x_2 &= 3 + 3x_4 + x_5 \end{aligned}$$

Faculty information

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